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Isoperimetric control of the spectrum of a compact hypersurface

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Abstract. Upper bounds for the eigenvalues of the Laplace–Beltrami operator on a hypersurface bounding a domain in some ambient Riemannian manifold are given in terms of the isoperimetric ratio of the domain. These results are applied to the extrinsic geometry of isometric embeddings.

1. Introduction

The spectrum of the Laplace–Beltrami operator on a compact Riemannian manifold (Σ, g) of dimension $n \geq 2$ provides a sequence of global Riemannian invariants

$$0 = \lambda_1(\Sigma) \leq \lambda_2(\Sigma) \leq \lambda_3(\Sigma) \leq \cdots \nearrow \infty.$$

One of the main goals of spectral geometry is to investigate relationships between these invariants and other geometric data of the manifold Σ such as the volume, the diameter, the curvature, or the Cheeger isoperimetric constant. See [2, 3, 7, 15] for classical references.

Since the work of Bleecker and Weiner, Reilly and others, the following approach has been developed: the manifold (Σ, g) is immersed isometrically into Euclidean space, or a more general ambient space. One then looks for relationships between the eigenvalues $\lambda_k(\Sigma)$ and extrinsic geometric quantities constructed from the second fundamental form of the immersed submanifold, such as the length of the mean curvature vector field. See for example [4, 16, 17, 21, 22, 25]. It is worth noticing that the spectrum of (Σ, g) cannot be controlled only by the volume of (Σ, g) (see [9, 11, 24]), even for isometrically embedded hypersurfaces (see [10, Theorem 1.4]).

More recently, the first two authors and E. Dryden [10] have obtained upper estimates for all normalized eigenvalues $\lambda_k(\Sigma)|\Sigma|^{2/n}$, where $|\Sigma|$ denotes the Riemannian volume of Σ, in terms of the number of intersection points of the immersed submanifold with a generic affine plane of complementary dimension. Such results allow a better understanding of the geometry

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of a Riemannian metric g on Σ inducing large eigenvalues, that is such that for some $k \geq 2$, the k-th normalized eigenvalue $\lambda_k(\Sigma, g)|(\Sigma, g)|^{2/n}$ is large. Indeed, if g is such a metric, then any isometric immersion of (Σ, g) into the Euclidean space $\mathbb{R}^{n+p}$ must have a large mean curvature, at least somewhere, and a large number of intersection points with some p-planes.

In the same vein, Reilly [25, Corollary 1] and Chavel [6] obtained the following remarkable inequality for the first positive eigenvalue $\lambda_2(\Sigma)$ in the case where Σ is embedded as a hypersurface bounding a domain Ω in $\mathbb{R}^{n+1}$ (or in a Cartan–Hadamard manifold in [6]):

$$\lambda_2(\Sigma)|\Sigma|^{\frac{2}{n}} \leq \frac{n}{(n+1)^2} I(\Omega)^{2+\frac{2}{n}}, \tag{1.1}$$

where $I(\Omega)$ is the isoperimetric ratio of Ω, that is

$$I(\Omega) = \frac{|\Sigma|}{|\Omega|^{\frac{n}{n+1}}},$$

where $|\Sigma|$ and $|\Omega|$ stand for the Riemannian n-volume of Σ and the Riemannian $(n+1)$-volume of Ω, respectively. Moreover, equality holds in (1.1) if and only if Σ is embedded as a round sphere.

The main feature of the upper bound (1.1) is its low sensitivity to small deformations, compared to that of the curvature or the intersection index. This result of Reilly and Chavel has been revisited by many authors [1, 18, 27], but only for the first non-zero eigenvalue λ_2, and using barycentric type methods involving coordinate functions.

Our aim in this paper is to establish inequalities of Reilly–Chavel type for higher order eigenvalues, that is to show that the isoperimetric ratio $I(\Omega)$ allows a control of the entire spectrum of $\Sigma = \partial\Omega$, and in various ambient spaces. Let us start with the particular but important case of compact hypersurfaces in Euclidean space.

Theorem 1.1. *For any bounded domain $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary $\Sigma = \partial\Omega$, and all $k \geq 1$,*

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \leq \gamma_n I(\Omega)^{1+\frac{2}{n}} k^{\frac{2}{n}}, \tag{1.2}$$

where $\gamma_n = \frac{2^{10n+18+8/n}}{n+1}\omega_{n+1}^{1/(n+1)}$ and ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$.

This result can also be understood as an estimate of the volume prescribed by a Riemannian manifold once embedded as an hypersurface in $\mathbb{R}^{n+1}$. That is, if (Σ, g) is a Riemannian manifold of dimension n of volume one, then, for any isometric embedding $\phi : \Sigma \to \mathbb{R}^{n+1}$, the domain Ω bounded by the hypersurface $\phi(\Sigma)$ satisfies, for each $k \geq 2$,

$$|\Omega|^{\frac{n+2}{n+1}} \leq \gamma_n \frac{k^{\frac{2}{n}}}{\lambda_k(\Sigma)}. \tag{1.3}$$

In particular, if the Riemannian metric g is such that λ_k is large, then the prescribed volume $|\Omega|$ has to be small (see [10, Theorem 1.4] for the existence of hypersurfaces with large λ_k).

For more general ambient spaces, we have the following theorem which is a particular case of a more general result (Theorem 2.1) we will prove in Section 2 in which the curvature assumptions are replaced by hypotheses of metric type.

Theorem 1.2. *Let (M,h) be a complete Riemannian manifold of dimension $n+1$ with Ricci curvature bounded below by $-na^2$, $a \in \mathbb{R}$. For any bounded domain $\Omega \subset M$ with smooth boundary $\Sigma = \partial\Omega$, and all $k \geq 1$, we have*

$$\lambda_k(\Sigma) \leq \alpha_n \frac{I(\Omega)}{I_0(\Omega)} a^2 + \beta_n \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n}} \left(\frac{k}{|\Sigma|}\right)^{\frac{2}{n}}, \tag{1.4}$$

where

$$I_0(\Omega) = \inf\{I(U) : U \textit{ is an open set in } \Omega\}$$

and α_n and β_n are two constants depending only on n (see (3.2) *and* (3.3) *for explicit expressions of these constants).*

Observe that the power of k appearing in the right-hand side of this estimate is optimal, according to Weyl's law.

It is in general not easy to estimate the number $I_0(\Omega)$, which represents the best constant in the isoperimetric inequality for domains in Ω. Recall that for any domain Ω in $\mathbb{R}^{n+1}$, one has

$$I_0(\Omega) = I_0(\mathbb{R}^{n+1}) = (n+1)\omega_{n+1}^{\frac{1}{n+1}},$$

where ω_{n+1} denotes the volume of the unit ball in $\mathbb{R}^{n+1}$. In a Cartan–Hadamard manifold, it is known that there exists a universal positive constant C_n such that $I(\Omega) \geq C_n$ for any bounded domain Ω (see [14]). More generally, if (M,h) is any complete Riemannian manifold with positive injectivity radius $\operatorname{inj}(M)$, then any domain U contained in a geodesic ball of radius $r < \frac{1}{2}\operatorname{inj}(M)$ satisfies $I(U) \geq C_n$ (see [14] and [8, Proposition V.2.3]). This leads to the following two corollaries.

Corollary 1.1. *Let (M,h) be a Cartan–Hadamard manifold of dimension $n+1$ with Ricci curvature bounded below by $-na^2$, $a \in \mathbb{R}$. For any bounded domain $\Omega \subset M$ with smooth boundary $\Sigma = \partial\Omega$, and all $k \geq 1$,*

$$\lambda_k(\Sigma) \leq A_n I(\Omega) a^2 + B_n I(\Omega)^{1+\frac{2}{n}} \left(\frac{k}{|\Sigma|}\right)^{\frac{2}{n}}, \tag{1.5}$$

where A_n and B_n are constants depending only on n.

In view of (1.1), it would be interesting to know if the first term on the right-hand side of inequality (1.5) is necessary. In Example 3.1 we will show that it is not always possible to remove this term, at least if we allow the topology of M to be non-trivial.

Corollary 1.2. *Let (M,h) be a complete Riemannian manifold of dimension $n+1$ with Ricci curvature bounded below by $-na^2$, $a \in \mathbb{R}$, and positive injectivity radius. For any compact hypersurface Σ bounding a domain $\Omega \subset M$ contained in a geodesic ball of radius $r < \frac{1}{2}\operatorname{inj}(M)$, and for each $k \geq 1$, one has*

$$\lambda_k(\Sigma) \leq A_n I(\Omega) a^2 + B_n I(\Omega)^{1+\frac{2}{n}} \left(\frac{k}{|\Sigma|}\right)^{\frac{2}{n}}, \tag{1.6}$$

where A_n and B_n are two constants depending only on n. In particular, for any bounded domain Ω in a hemisphere of the standard sphere $\mathbb{S}^{n+1}$ with smooth boundary $\Sigma = \partial\Omega$, and all $k \geq 1$,

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \leq B_n I(\Omega)^{1+\frac{2}{n}} k^{\frac{2}{n}}.$$

The assumption that the domain is contained in a geodesic ball of radius $r < \frac{1}{2}\,\mathrm{inj}(M)$ is necessary. Indeed, in Example 3.2 below, we will show that if (M, h) is any compact manifold, then there exists a sequence of domains for which inequality (1.6) fails, whatever the constants A_n and B_n are.

Notice that it is impossible to obtain an inequality such as (1.6) for a class of domains Ω in a Riemannian manifold M without an assumption that guarantees that their isoperimetric ratio $I(\Omega)$ is uniformly bounded from below. Indeed, since $\lambda_k(\Sigma)|\Sigma|^{2/n} \sim c_n k^{2/n}$ as $k \to \infty$ (Weyl's asymptotic formula with $c_n = 4\pi^2\omega_n^{-2/n}$), the inequality (1.6) implies that

$$I(\Omega)^{1+\frac{2}{n}} \geq c_n/B_n.$$

Finally, let us mention that in our recent work [12], we studied isoperimetric control of the Steklov spectrum for bounded domains in a complete Riemannian manifold. The methods we used in [12] are based on concentration properties which were initiated by Korevaar [23], and further developed by Grigor'yan, Netrusov and Yau [19, 20]. Together with the results of the present paper, this leads to comparison results between the Steklov spectrum of a domain and the spectrum of its boundary hypersurface. See [12, Section 4] for details.

2. Eigenvalue bounds: a general result

In this section, we give an upper bound for the eigenvalues of the Laplacian in terms of quantities which depend only on the Riemannian distance and measure.

Let M be a Riemannian manifold M of dimension $n + 1$. The Riemannian volumes of a geodesic ball $B(x, r)$ and of a geodesic sphere $\partial B(x, r)$ of radius r in M are asymptotically equivalent as $r \to 0$ to $\omega_{n+1} r^{n+1}$ and $\rho_n r^n$, respectively, where ω_{n+1} is the volume of a unit ball and $\rho_n = (n + 1)\omega_{n+1}$ is the volume of a unit sphere in the $(n + 1)$-dimensional Euclidean space. To each point x in M we associate the number $r(x)$ defined as the largest positive number (possibly infinite) so that, for all $r < r(x)$, one has

$$|B(x, r)| < 2\omega_{n+1} r^{n+1}$$

and

$$|\partial B(x, r)| < 2\rho_n r^n.$$

If M has nonnegative Ricci curvature, then, thanks to the Bishop–Gromov inequality, one has $r(x) = +\infty$ for all $x \in M$.

Let Ω be a bounded regular domain in M and denote by Σ the boundary of Ω. We define the number $r_-(\Omega)$ as follows:

$$r_-(\Omega) = \inf_{x\in\Sigma} r(x).$$

We also introduce, for all $r > 0$, an integer $N_M(r)$ such that, for any $x \in M$ and any $s < r$, the geodesic ball $B(x, 4s)$ can be covered by $N_M(r)$ balls of radius s.

The main technical result of this paper is the following:

Proposition 2.1. *Let r_0 be a positive number such that $r_0 < \frac{1}{4}r_-(\Omega)$ and define k_0 to be the first integer satisfying*

$$k_0 > \frac{1}{16\rho_n}\frac{I_0(\Omega)}{r_0^n}|\Omega|^{\frac{n}{n+1}}.$$

For all $k \geq k_0$,

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \leq 256\,(16\rho_n)^{\frac{2}{n}} N_M(r_0)^2 \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n}} k^{\frac{2}{n}}.$$

Proposition 2.1 has the following consequence, from which the results announced in the introduction will follow.

Theorem 2.1. *Let M be a complete Riemannian manifold of dimension $n+1$ and let $\Omega \subset M$ be a bounded domain whose boundary Σ is a smooth hypersurface. For any $r_0 < \frac{1}{4}r_-(\Omega)$ and any positive integer k, one has*

$$\lambda_k(\Sigma) \leq 256\, N_M(r_0)^2 \frac{I(\Omega)}{I_0(\Omega)}\left\{\frac{1}{r_0^2} + \left(16\rho_n \frac{I(\Omega)}{I_0(\Omega)}\frac{k}{|\Sigma|}\right)^{\frac{2}{n}}\right\}. \tag{2.1}$$

It is in general not easy to estimate the quantities $I_0(\Omega)$ and $N_M(r_0)$ that appear in the right-hand side of this inequality. However, in many standard geometric situations it is possible to control these invariants in terms of the dimension and a lower bound of the Ricci curvature. This will lead to the results stated in the introduction. For example, when M is the Euclidean space $\mathbb{R}^{n+1}$, then, for any $\Omega \subset \mathbb{R}^{n+1}$, one has $r_-(\Omega) = +\infty$,

$$I_0(\Omega) = I_0(\mathbb{R}^{n+1}) = (n+1)\omega_{n+1}^{\frac{1}{n+1}}$$

and $N_M(r) \leq 32^{n+1}$ for all $r > 0$ (see Lemma 3.1 below).

For the need of the proof, we endow M with the Borel measure μ with support in Σ defined for each Borelian $\mathcal{O} \subset M$ by

$$\mu(\mathcal{O}) = \int_{\mathcal{O}\cap\Sigma} dv_g,$$

In other words, the μ-measure of $\mathcal{O}$ is the volume of the part of the hypersurface Σ lying inside $\mathcal{O}$. The geodesic distance of M will be denoted by d.

One of the main tools in the proof is the following result which is an adapted version of a result obtained by Maerten and the first author in [13]:

Lemma 2.1. *Let (X, d, μ) be a complete, locally compact metric measure space, where μ is a finite measure. We assume that, for all $r > 0$, there exists an integer $N(r)$ such that each ball of radius $4r$ can be covered by $N(r)$ balls of radius r. If there exist an integer $K > 0$ and a radius $r > 0$ such that, for each $x \in X$,*

$$\mu(B(x,r)) \leq \frac{\mu(X)}{4N^2(r)K},$$

then there exist K μ-measurable subsets $A_1, \dots, A_K$ of X such that

$$\mu(A_i) \geq \frac{\mu(X)}{2N(r)K} \quad \text{for all } i \leq K,$$

and $d(A_i, A_j) \geq 3r$ for $i \neq j$.

The proof of this lemma consists of a slight modification of the construction made in [13, Section 2]. For convenience, the proof is included at the end of the paper.

Proof of Proposition 2.1. The Rayleigh quotient of a function f in the Sobolev space $H^1(\Sigma)$ is

$$R(f) = \frac{\int_\Sigma |\nabla_\Sigma f|^2}{\int_\Sigma f^2}.$$

The k-th eigenvalue $\lambda_k(\Sigma)$ is characterized as follows:

$$\lambda_k(\Sigma) = \inf_E \sup_{0 \neq f \in E} R(f),$$

where the infimum is over all k-dimensional subspaces of the Sobolev space $H^1(\Sigma)$ (see for instance [2]). In particular, in order to obtain upper bounds on λ_k, we will construct k test functions with disjoint supports and controlled Rayleigh quotient.

Let us fix an integer $k \geq k_0$ and set

$$r_k = \left(\frac{I_0(\Omega)}{4^{n+2}\rho_n k}\right)^{\frac{1}{n}} |\Omega|^{\frac{1}{n+1}} \tag{2.2}$$

so that

$$r_k^n \leq \frac{1}{4^n} \frac{I_0(\Omega)|\Omega|^{\frac{n}{n+1}}}{16\rho_n k_0} < \left(\frac{r_0}{4}\right)^n,$$

that is $r_k < \frac{r_0}{4}$.

Step 1. Let us first show that Σ cannot be covered by $2k$ balls of radius $4r_k$. More precisely, let $x_1, x_2, \dots, x_{2k}$ be $2k$ (arbitrary) points in M and define

$$M_0 = M \setminus \bigcup_{j=1}^{2k} B(x_j, 4r_k),$$

$$\Omega_0 = \Omega \setminus \bigcup_{j=1}^{2k} B(x_j, 4r_k), \qquad \Sigma_0 = \Sigma \setminus \bigcup_{j=1}^{2k} B(x_j, 4r_k).$$

Then,

$$|\Omega_0| > \frac{3}{4}|\Omega| \tag{2.3}$$

and

$$|\Sigma_0| > \frac{1}{2} I_0(\Omega)|\Omega|^{\frac{n}{n+1}} = \frac{1}{2}\frac{I_0(\Omega)}{I(\Omega)}|\Sigma|. \tag{2.4}$$

Indeed, since $4r_k < r_0 < r_-(\Omega)$, one has

$$\sum_{j=1}^{2k} |B(x_j, 4r_k)| < 4k\omega_{n+1}(4r_k)^{n+1}$$

with

$$(4r_k)^{n+1} = \left(\frac{I_0(\Omega)}{16\rho_n k}\right)^{\frac{n+1}{n}} |\Omega| < \frac{1}{16k}\left(\frac{I_0(\Omega)}{\rho_n}\right)^{\frac{n+1}{n}} |\Omega| \le \frac{1}{16k\omega_{n+1}}|\Omega|,$$

where the last inequality follows from the fact that

$$I_0(\Omega) \le I_0(\mathbb{R}^{n+1}) = \frac{\rho_n}{\omega_{n+1}^{\frac{n}{n+1}}}.$$

Therefore,

$$\sum_{j=1}^{2k} |B(x_j, 4r_k)| < \frac{1}{4}|\Omega|$$

and

$$|\Omega_0| > |\Omega| - \frac{1}{4}|\Omega| = \frac{3}{4}|\Omega|.$$

Now, observe that the boundary of Ω_0 consists of the union of Σ_0 and parts of the boundaries of the balls $B(x_j, 4r_k)$. Therefore,

$$|\partial\Omega_0| \le |\Sigma_0| + \sum_{j=1}^{2k} |\partial B(x_j, 4r_k)| < |\Sigma_0| + 4k\rho_n(4r_k)^n = |\Sigma_0| + \frac{1}{4} I_0(\Omega)|\Omega|^{\frac{n}{n+1}}.$$

On the other hand, from the isoperimetric inequality satisfied by domains in Ω and (2.3) we get

$$|\partial\Omega_0| \ge I_0(\Omega)|\Omega_0|^{\frac{n}{n+1}} > \left(\frac{3}{4}\right)^{\frac{n}{n+1}} I_0(\Omega)|\Omega|^{\frac{n}{n+1}}.$$

Hence,

$$|\Sigma_0| > \left[\left(\frac{3}{4}\right)^{\frac{n}{n+1}} - \frac{1}{4}\right] I_0(\Omega)|\Omega|^{\frac{n}{n+1}} > \frac{1}{2} I_0(\Omega)|\Omega|^{\frac{n}{n+1}}.$$

Step 2. The result of the previous step makes it possible to define inductively a family of $2k$ balls $B(x_1, r_k), \dots, B(x_{2k}, r_k)$ satisfying the following:

$$\mu(B(x_1, r_k)) = \sup_{x \in M} \mu(B(x, r_k)),$$

$$\mu(B(x_{j+1}, r_k)) = \sup\Big\{\mu(B(x, r_k)) \,:\, x \in M \setminus \bigcup_{i=1}^{j} B(x_i, 4r_k)\Big\}.$$

It follows from this construction that

(a) the balls $B(x_1, 2r_k), \dots, B(x_{2k}, 2r_k)$ are mutually disjoint,

(b) $\mu(B(x_1, r_k)) \geq \mu(B(x_2, r_k)) \geq \cdots \geq \mu(B(x_{2k}, r_k))$,

(c) for all $x \in M_0 = M \setminus \bigcup_{j=1}^{2k} B(x_j, 4r_k)$, one has $\mu(B(x, r_k)) \leq \mu(B(x_{2k}, r_k))$.

Two alternatives are to be considered separately, depending on how the ball $B(x_{2k}, r_k)$ is μ-charged. This will be done in the two following steps.

Step 3. Assuming that

$$\mu(B(x_{2k}, r_k)) \geq \frac{I_0(\Omega)|\Omega|^{\frac{n}{n+1}}}{16kN_M(r_0)^2} = \frac{1}{16kN_M(r_0)^2}\frac{I_0(\Omega)}{I(\Omega)}|\Sigma|, \tag{2.5}$$

we show that

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \leq \frac{16N_M(r_0)^2}{r_k^2}\frac{I(\Omega)}{I_0(\Omega)}|\Sigma|^{\frac{2}{n}}.$$

Indeed, for each $1 \leq j \leq 2k$, we consider the function f_j supported in $B(x_j, 2r_k)$ and defined, for all $x \in B(x_j, 2r_k)$, by

$$f_j(x) = \min\left\{1,\ 2 - \frac{1}{r_k}d(x_j, x)\right\}. \tag{2.6}$$

Since $|\nabla f_j|^2 \leq \frac{1}{r_k^2}$ in $B(x_j, 2r_k)$, the Rayleigh quotient of the restriction of f_j to Σ, that we still denote by f_j, clearly satisfies

$$R(f_j) \leq \frac{1}{r_k^2}\frac{\mu(B(x_j, 2r_k))}{\mu(B(x_j, r_k))} \tag{2.7}$$

with (from the definition of $x_1, \dots, x_{2k}$)

$$\mu(B(x_j, r_k)) \geq \mu(B(x_{2k}, r_k)) \geq \frac{1}{16kN_M(r_0)^2}\frac{I_0(\Omega)}{I(\Omega)}|\Sigma|.$$

On the other hand, the balls $B(x_j, 2r_k)$, $j = 1, \dots, 2k$, being mutually disjoint, there exist k of them, $B(x_{j_1}, 2r_k), \dots, B(x_{j_k}, 2r_k)$ satisfying

$$\mu(B(x_{j_m}, 2r_k)) \leq \frac{|\Sigma|}{k} \quad \text{for } m = 1, \dots, k.$$

Together with (2.7) this yields, for all $m = 1, \dots, k$,

$$R(f_{j_m}) < \frac{16N_M(r_0)^2}{r_k^2}\frac{I(\Omega)}{I_0(\Omega)}$$

so that

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \leq \max_{1\leq m\leq k} R(f_{j_m})|\Sigma|^{\frac{2}{n}} \leq \frac{16N_M(r_0)^2}{r_k^2}\frac{I(\Omega)}{I_0(\Omega)}|\Sigma|^{\frac{2}{n}}.$$

Step 4. Assuming now that

$$\mu(B(x_{2k}, r_k)) < \frac{1}{16kN_M(r_0)^2} \frac{I_0(\Omega)}{I(\Omega)} |\Sigma|, \tag{2.8}$$

we show that

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \le \frac{8N_M(r_0)}{r_k^2} \frac{I(\Omega)}{I_0(\Omega)} |\Sigma|^{\frac{2}{n}}.$$

Indeed, from the construction of the balls $B(x_j, r_k)$ (see step 2), one has that, for all $x \in M_0 = M \setminus \bigcup_{j=1}^{2k} B(x_j, 4r_k)$,

$$\mu(B(x, r_k)) \le \mu(B(x_{2k}, r_k)) < \frac{1}{16kN_M(r_0)^2} \frac{I_0(\Omega)}{I(\Omega)} |\Sigma|$$

with $\mu(\Sigma_0) = |\Sigma_0| > \frac{1}{2}\frac{I_0(\Omega)}{I(\Omega)}|\Sigma|$ (see (2.4)). Hence, for all $x \in M_0$, we have

$$4N_M(r_0)^2\mu(B(x, r_k)) < \frac{\mu(\Sigma_0)}{2k}. \tag{2.9}$$

This enables us to apply Lemma 2.1 with $K = 2k$ and $r = r_k$ to the metric measure space M endowed with the Riemannian distance d and the restriction μ_0 of the measure μ to Σ_0, namely for a Borelian $\mathcal{O} \subset M$, we have $\mu_0(\mathcal{O}) = \mu(\mathcal{O} \cap \Sigma_0)$. In particular,

$$\mu_0(M) = \mu(\Sigma_0) = |\Sigma_0| > \frac{1}{2} \frac{I_0(\Omega)}{I(\Omega)} |\Sigma|. \tag{2.10}$$

The relation (2.9) becomes

$$4N_M(r_0)^2\mu_0(B(x, r_k)) < \frac{\mu_0(M)}{2k}. \tag{2.11}$$

Thus, we deduce the existence of $2k$ measurable sets $A_1, \dots, A_{2k}$ in M_0 satisfying both:

$$\mu(A_i) \ge \frac{\mu_0(\Sigma_0)}{4kN_M(r_0)} \quad \text{for all } i = 1, \dots, 2k$$

and $d(A_i, A_j) \ge 3r_k$ if $i \ne j$. Denote by

$$A_i^{r_k} = \{x \in M : d(x, A_i) < r_k\}$$

the r_k-neighborhood of A_i. A priori, we have no control over $\mu_0(A_i^{r_k})$, but since we have $d(A_i, A_j) \ge 3r$ for $i \ne j$, the $A_i^{r_k}$ are mutually disjoint and there exist k sets amongst them, say $A_1^{r_k}, \dots, A_k^{r_k}$, which satisfy

$$\mu_0(A_i^{r_k}) \le \frac{|\Sigma|}{k} \quad \text{for } i = 1, \dots, k.$$

As in [13], we construct for each $i \le k$ a test function φ_i with support in $A_i^{r_k}$ and which is defined, for all $x \in A_i^{r_k}$, by

$$\varphi_i(x) = 1 - \frac{d(x, A_i)}{r_k}.$$

Observing that $|\nabla\varphi_i(x)| \le \frac{1}{r_k}$ almost everywhere in $A_i^{r_k}$, a straightforward calculation shows that the Rayleigh quotient of the restriction of φ_i to Σ, that we still denote by φ_i, satisfies

$$R(\varphi_i) \le \frac{1}{r_k^2}\,\frac{\mu_0(A_i^r)}{\mu_0(A_i)} < \frac{1}{r_k^2}\,\frac{|\Sigma|}{\frac{\mu_0(M)}{4N_M(r_0)}}$$

and, because of (2.10), we have

$$R(\varphi_i) \le \frac{8N_M(r_0)}{r_k^2}\,\frac{I(\Omega)}{I_0(\Omega)}.$$

Thus,

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \le \max_{1\le i\le k} R(\varphi_i)|\Sigma|^{\frac{2}{n}} \le \frac{8N_M(r_0)}{r_k^2}\,\frac{I(\Omega)}{I_0(\Omega)}\,|\Sigma|^{\frac{2}{n}}.$$

Step 5. We are now ready to conclude the proof. From the two previous steps, we see that in all cases, one has

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \le \frac{16N_M(r_0)^2}{r_k^2}\,\frac{I(\Omega)}{I_0(\Omega)}|\Sigma|^{\frac{2}{n}}$$

with

$$\frac{|\Sigma|^{\frac{2}{n}}}{r_k^2} = \left(\frac{4^{n+2}\rho_n k}{I_0(\Omega)}\right)^{\frac{2}{n}}\frac{|\Sigma|^{\frac{2}{n}}}{\Omega^{\frac{2}{n+1}}} = (4^{n+2}\rho_n k)^{\frac{2}{n}}\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{\frac{2}{n}}.$$

Thus,

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \le 256\,(16\rho_n)^{\frac{2}{n}}\,N_M(r_0)^2\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n}} k^{\frac{2}{n}}. \qquad \square$$

Proof of Theorem 2.1. Let k be a positive integer. If $k < k_0$, then $\lambda_k(\Sigma) \le \lambda_{k_0}(\Sigma)$. Together with Proposition 2.1, this yields, for all $k \ge 1$,

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \le 256\,(16\rho_n)^{\frac{2}{n}}\,N_M(r_0)^2\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n}} \max\left\{k_0^{\frac{2}{n}}, k^{\frac{2}{n}}\right\}.$$

We clearly have $\max\{k_0^{2/n}, k^{2/n}\} \le (k_0-1)^{2/n}+k^{2/n}$, with $k_0-1 \le \frac{1}{16\rho_n}\frac{I_0(\Omega)}{r_0^n}|\Omega|^{n/(n+1)}$. Consequently

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \le 256\,N_M(r_0)^2\left\{\frac{I(\Omega)^{1+\frac{2}{n}}}{I_0(\Omega)}\,\frac{|\Omega|^{\frac{2}{n+1}}}{r_0^2} + (16\rho_n)^{\frac{2}{n}}\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n}} k^{\frac{2}{n}}\right\}.$$

Replacing $|\Omega|^{2/(n+1)}$ by $\frac{|\Sigma|^{2/n}}{I(\Omega)^{2/n}}$, we get

$$\lambda_k(\Sigma) \le 256\,N_M(r_0)^2\frac{I(\Omega)}{I_0(\Omega)}\left\{\frac{1}{r_0^2} + \left(16\rho_n\frac{I(\Omega)}{I_0(\Omega)}\,\frac{k}{|\Sigma|}\right)^{\frac{2}{n}}\right\}. \qquad \square$$

3. Proof of Theorems 1.1 and 1.2 and comments

The proofs of these theorems rely on Bishop–Gromov comparison results and the following packing lemma (see [28, Lemma 3.6]).

Lemma 3.1. *Let (X, d, ν) be a locally compact metric measure space and let r, R and V be positive numbers with $r < R$ and such that, for all $x \in X$,*

$$\frac{\nu(B(x, 2R))}{\nu(B(x, \frac{r}{4}))} \leq V.$$

Then each ball of radius R in X can be covered by $\lfloor V \rfloor$ balls of radius r. In particular, when the ambient space is the standard $\mathbb{R}^{n+1}$, then any ball of radius R can be covered by $(8\frac{R}{r})^{n+1}$ balls of radius r.

Proof of Theorem 1.1. Let Ω be a bounded domain in $\mathbb{R}^{n+1}$. With the notations of the last section, one clearly has $r_-(\Omega) = \infty$,

$$I_0(\Omega) = I_0(\mathbb{R}^{n+1}) = (n+1)\omega_{n+1}^{\frac{1}{n+1}}$$

and $N_M(r) \leq 32^{n+1}$ for all $r > 0$ by Lemma 3.1. We then apply Theorem 2.1 to get, after letting r_0 go to infinity, for all $k \geq 1$,

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \leq \gamma_n I(\Omega)^{1+\frac{2}{n}} k^{\frac{2}{n}} \quad \text{with } \gamma_n = \frac{2^{10n+18+\frac{8}{n}}}{(n+1)}\, \omega_{n+1}^{\frac{1}{n+1}}. \qquad \square$$

Proof of Theorem 1.2. Let (M, h) be a complete Riemannian manifold of dimension $n+1$ whose Ricci curvature tensor satisfies

$$\mathrm{Ric} \geq -na^2 h$$

for some constant a. Let Ω be a bounded domain in M with regular boundary $\Sigma = \partial\Omega$. To prove Theorem 1.2, we treat separately the three following cases:

Case $a = 0$. That is, M has nonnegative Ricci curvature. From Bishop–Gromov comparison results (see [26, p. 156]) we deduce that, for all $x \in M$ and all $r > 0$,

$$|B(x,r)| \leq \omega_{n+1} r^{n+1}, \qquad |\partial B(x,r)| \leq \rho_n r^n, \qquad \frac{|B(x, 8r)|}{|B(x, \frac{r}{4})|} \leq 32^{n+1}.$$

Thus, $r_-(\Omega) = \infty$ and, applying Lemma 3.1, $N_M(r) \leq 32^{n+1}$ for all $r > 0$. Substituting in (2.1) and letting r_0 go to infinity we get, for all $k \geq 1$,

$$\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \leq 2^{10(n+1)+8}(16\rho_n)^{\frac{2}{n}} \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n}} k^{\frac{2}{n}}.$$

Case $a = 1$. The volumes of a ball and of a sphere of radius r in the hyperbolic space $\mathbb{H}^{n+1}$ of curvature -1 and dimension $n+1$ are given by

$$V_{-1}(n,r) = \rho_n \int_0^r (\sinh s)^n ds \quad \text{and} \quad S_{-1}(n,r) = \rho_n (\sinh r)^n.$$

We define the constant $r(n)$ to be the largest $r > 0$ such that $(\sinh r)^n \le 2r^n$ and set

$$V(n) = \sup_{0<r<r(n)} \frac{V_{-1}(n,8r)}{V_{-1}(n,\frac{r}{4})}.$$

Again, the Bishop–Gromov comparison theorem gives, for all $x \in M$ and all positive $r < r(n)$,

$$|B(x,r)| \le V_{-1}(n,r) < 2\omega_{n+1} r^{n+1}, \qquad |\partial B(x,r)| \le S_{-1}(n,r) < 2\rho_n r^n,$$
$$\frac{|B(x,8r)|}{|B(x,\frac{r}{4})|} \le \frac{V_{-1}(n,8r)}{V_{-1}(n,\frac{r}{4})} \le V(n).$$

Thus, $r_-(\Omega) \ge r(n) > 0$ and, applying Lemma 3.1, $N_M(r) \le V(n)$ for all $r < r(n)$. Applying Theorem 2.1 we get

$$\lambda_k(\Sigma) \le 256\, V^2(n) \frac{I(\Omega)}{I_0(\Omega)} \left\{ \frac{1}{r^2(n)} + \left(16\rho_n \frac{I(\Omega)}{I_0(\Omega)} \frac{k}{|\Sigma|} \right)^{\frac{2}{n}} \right\},$$

that is

$$\lambda_k(\Sigma) \le \alpha_n \frac{I(\Omega)}{I_0(\Omega)} + \beta_n \left(\frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n}} \left(\frac{k}{|\Sigma|} \right)^{\frac{2}{n}}, \tag{3.1}$$

with $\alpha_n = 256 \frac{V^2(n)}{r^2(n)}$ and $\beta_n = (16\rho_n)^{2/n} \alpha_n$.

Case $a \ne 0$. The metric $\tilde{h} = a^2 h$ is such that $\mathrm{Ric}_{\tilde{h}} \ge -n\tilde{h}$. The metric $\tilde{g} = a^2 g$ induced on Σ by $\tilde{h}$ is so that $\lambda_k(\Sigma,\tilde{g}) = \frac{1}{a^2}\lambda_k(\Sigma)$ and $|(\Sigma,\tilde{g})| = a^n |\Sigma|$ while the isoperimetric ratio is invariant under scaling. Thus, applying the inequality (3.1) to Ω considered as a domain in $(M,\tilde{h})$ we get

$$\frac{1}{a^2}\lambda_k(\Sigma) \le \alpha_n \frac{I(\Omega)}{I_0(\Omega)} + \beta_n \left(\frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n}} \left(\frac{k}{a^n|\Sigma|} \right)^{\frac{2}{n}},$$

which gives, after simplification, the desired inequality.

In conclusion, inequality (1.4) is proved with

$$\alpha_n = 0 \quad \text{and} \quad \beta_n = 2^{10(n+1)+8} (16\rho_n)^{\frac{2}{n}} \tag{3.2}$$

if $a = 0$, and

$$\alpha_n = 256 \frac{V^2(n)}{r^2(n)} \quad \text{and} \quad \beta_n = (16\rho_n)^{\frac{2}{n}} \alpha_n \tag{3.3}$$

if $a \ne 0$. □

Example 3.1. The aim of the following construction is to show that the boundedness of the sectional curvature does not suffice to get an estimate such as (1.2). Indeed, we will construct a sequence of manifolds (M_i, h_i) whose sectional curvature is between -1 and 0, each containing a domain Ω_i with boundary Σ_i such that $I(\Omega_i) = 1$, $|\Sigma_i|$ tends to infinity with i, and $\lambda_3(\Sigma_i)$ is bounded below by a positive constant.

According to [5], there exists a sequence N_i of compact hyperbolic manifolds of dimension $n \geq 2$ whose volume tends to infinity with i while $\lambda_2(N_i)$ does not converge to zero. We can assume that $|N_i| > i$ and $\lambda_2(N_i) > C$ for some positive constant C. For each i, set $M_i = N_i \times \mathbb{R}$ and $\Omega_i = N_i \times (-L_i, L_i) \subset M_i$ with $L_i = (2|N_i|)^{1/n}$. We endow M_i with the product metric, so that the sectional curvature of M_i is between -1 and zero. The boundary Σ_i of the domain Ω_i consists of two disjoint copies of N_i. Therefore, $\lambda_1(\Sigma_i) = \lambda_2(\Sigma_i) = 0$ and $\lambda_3(\Sigma_i) = \lambda_2(N_i) > C$. On the other hand, we have

$$|\Sigma_i| = 2|N_i| > 2i, \qquad |\Omega_i| = 2L_i|N_i| = (2|N_i|)^{\frac{n+1}{n}}, \qquad I(\Omega_i) = 1.$$

Example 3.2. Let (M, h) be any compact Riemannian manifold of dimension $n+1 \geq 3$. Then there exists a sequence Ω_i of domains in M with smooth boundaries Σ_i, and a positive constant C such that $\lambda_2(\Sigma_i)|\Sigma_i|^{2/n} \geq C$ while $|\Sigma_i|$ and $I(\Omega_i)$ go to zero as i tends to infinity.

In particular, there exist no constants A_n and B_n such that the sequence Ω_i satisfies an inequality like (1.6).

Let us first assume that (M, h) is flat in a geodesic ball $B(x_0, r)$ centered at some $x_0 \in M$. This ball is isometric to a Euclidean ball of radius r. Let S_i be a sequence of Euclidean spheres of radius $\frac{r}{i}$ that we embed isometrically as hypersurfaces Σ_i into $B(x_0, r) \subset M$. For each i, the hypersurface Σ_i bounds a domain Ω_i which contains the complement of $B(x_0, r)$. This sequence of domains Ω_i satisfies

$$|\Omega_i| = |M| - \omega_{n+1}\left(\frac{r}{i}\right)^{n+1}, \qquad |\Sigma_i| = \rho_n\left(\frac{r}{i}\right)^{n},$$

that is, $|\Sigma_i|$ and $I(\Omega_i)$ go to zero as i tends to infinity. On the other hand,

$$\lambda_2(\Sigma_i)|\Sigma_i|^{\frac{2}{n}} = \lambda_2(S_i)|S_i|^{\frac{2}{n}} = n\rho_n^{\frac{2}{n}}.$$

Now, for a general compact Riemannian manifold (M, h), it is possible to deform the metric h into a metric h' which is quasi-isometric to h with quasi-isometry ratio close to 1, and so that (M, h') is flat in a small geodesic ball $B(x_0, r)$. The sequence Ω_i of domains constructed above with respect to h' would be such that, for the metric h, $|\Sigma_i|$ and $I(\Omega_i)$ go to zero as i tends to infinity while $\lambda_2(\Sigma_i)|\Sigma_i|^{2/n}$ is bounded below by a positive constant.

4. Proof of Lemma 2.1

Let (X, d, μ) be a complete, locally compact metric measure space, where μ is a finite measure. We assume that, for all $r > 0$, there exists an integer $N(r)$ such that each ball of radius $4r$ can be covered by $N(r)$ balls of radius r. Let us first prove the following result.

Lemma 4.1. *Let β be a positive number satisfying $\beta \leq \frac{\mu(X)}{2}$, and let $r > 0$ be such that, for all $x \in X$,*

$$\mu(B(x,r)) \leq \frac{\beta}{2N(r)}.$$

Then there exist two open subsets A and D of X with $A \subset D$, such that

$$\mu(A) \geq \beta, \qquad \mu(D) \leq 2N(r)\beta, \qquad d(A, D^c) \geq 3r.$$

Proof. For each positive integer m, we denote by $\mathcal{U}_m(r)$ the set of unions of m balls of radius r, that is,

$$\mathcal{U}_m(r) := \left\{ \bigcup_{j=1}^{m} B(x^j, r) \;:\; x^1, \ldots, x^m \in X \right\},$$

and consider the evaluation Ψ_m of the measure μ on $\mathcal{U}_m(r)$, that is

$$\Psi_m : X^m = \underbrace{X \times X \times \cdots \times X}_{m \text{ times}} \longrightarrow \mathbb{R}$$

with

$$\Psi_m(x^1, \ldots, x^m) = \mu\left(\bigcup_{j=1}^{m} B(x^j, r) \right).$$

Since (X, d) is a complete locally compact metric space and $\mu(X) < +\infty$, the function Ψ_m achieves its maximum $\xi(m)$ at some point $\mathbf{a}_m = (a_m^1, \ldots, a_m^m) \in X^m$ (not necessary unique), that is,

$$\mu\left(\bigcup_{j=1}^{m} B(x^j, r) \right) \leq \mu\left(\bigcup_{j=1}^{m} B(a_m^j, r) \right)$$

for any $(x^1, \ldots, x^m) \in X^m$.

Now, from the assumptions of the lemma one clearly has $\xi(1) \leq \frac{\beta}{2N(r)} \leq \beta$. On the other hand, for m large enough, we necessarily have $\xi(m) \geq \frac{3\beta}{2}$ (indeed, it suffices to consider a ball $B(z, R)$ satisfying $\mu(B(z, R)) \geq \frac{3}{4}\mu(X)$ and notice that it can be covered with a finite number of balls of radius r). In conclusion, there exists an integer $k \geq 2$ such that $\xi(k) \geq \beta$ and $\xi(k-1) \leq \beta$.

We set

$$A := \bigcup_{1 \leq j \leq k} B(a_k^j, r) \quad \text{and} \quad D := \bigcup_{1 \leq j \leq k} B(a_k^j, 4r).$$

From their definitions, these sets satisfy $\mu(A) = \xi(k) \geq \beta$ and $d(A, D^c) \geq 3r$. We still need to check that $\mu(D) \leq 2N(r)\beta$. Indeed, according to our hypotheses, each ball $B(a_k^j, 4r)$ can be covered by $N(r)$ balls of radius r. Hence, D can be covered by $kN(r)$ balls of radius r, namely $D \subset \bigcup_{1 \leq j \leq kN(r)} B_j$, where the B_j are balls of radius r. From $kN(r) \leq 2(k-1)N(r)$, it follows that this union of balls can be written as

$$\bigcup_{1 \leq j \leq kN(r)} B_j \;=\; \bigcup_{1 \leq j \leq 2N(r)} W_j,$$

where each $W_j \in \mathcal{U}_{k-1}(r)$. It follows that

$$\mu(D) \leq \mu\left(\bigcup_{j=1}^{2N(r)} W_j \right) \leq \sum_{j=1}^{2N(4r)} \mu(W_j) \leq 2N(r)\xi(k-1) \leq 2N(r)\beta. \qquad \square$$

Proof of Lemma 2.1. Let $K \geq 2$ be an integer and $r > 0$ a positive number such that, for all $x \in X$,

$$\mu(B(x,r)) \leq \frac{\mu(X)}{4N^2(r)K}.$$

Our aim is to construct K μ-measurable subsets $A_1, \dots, A_K$ of X such that $\mu(A_i) \geq \frac{\mu(X)}{2N(r)K}$ for all $i \leq K$, and $d(A_i, A_j) \geq 3r$ for $i \neq j$.

For simplicity, we set $\alpha = \frac{\mu(X)}{2N(r)K}$. We shall construct, using a finite induction, K pairs $(A_1, D_1), \dots, (A_K, D_K)$ of sets such that, for all $j \leq K$,

(0) $A_j \subset D_j$,

(1) $A_j \subset \big(\bigcup_{i=1}^{j-1} D_i\big)^c$,

(2) $\mu(A_j) \geq \alpha$,

(3) $\mu(D_j) \leq 2N(r)\alpha = \frac{\mu(X)}{K}$,

(4) $d\big(A_j, \big(\bigcup_{i=1}^{j} D_i\big)^c\big) \geq 3r$.

Indeed, the family $A_1, \dots, A_K$ will then satisfy the desired properties since $\mu(A_j) \geq \alpha$ and, if $k < j$,

$$d(A_k, A_j) \geq d\left(A_k, \left(\bigcup_{i=1}^{k} D_i\right)^c\right) \geq 3r.$$

(Notice that $A_j \subset \big(\bigcup_{i=1}^{j-1} D_i\big)^c \subset \big(\bigcup_{i=1}^{k} D_i\big)^c$ since $k < j$.)

To initiate the iteration it suffices to apply Lemma 4.1 with $\beta = \alpha$. Therefore, there exist two open sets A_1 and D_1 satisfying $A_1 \subset D_1$ and

$$\mu(A_1) \geq \alpha, \qquad \mu(D_1) \leq 2N(r)\alpha = \frac{\mu(X)}{K}, \qquad d(A_1, D_1^c) \geq 3r.$$

Now, let us assume that we have already constructed, for a certain $j < K$, j couples $(A_1, D_1), \dots, (A_j, D_j)$ satisfying the induction hypothesis. We endow X with the measure μ_{j+1} defined by

$$\mu_{j+1}(U) = \mu\left(U \cap \left(\bigcup_{i=1}^{j} D_i\right)^c\right).$$

From the induction hypothesis, one has

$$\mu_{j+1}(X) = \mu\left(\left(\bigcup_{i=1}^{j} D_i\right)^c\right) \geq \mu(X) - \sum_{i=1}^{j} \mu(D_i) \geq \mu(X)\left(1 - \frac{j}{K}\right) \geq \frac{\mu(X)}{K}.$$

Therefore, for all $x \in X$, one has

$$\mu_{j+1}(B(x,r)) \leq \mu(B(x,r)) \leq \frac{\mu(X)}{4N^2(r)K} \leq \frac{\mu_{j+1}(X)}{4N^2(r)}.$$

This allows us to apply Lemma 4.1 to the metric measure space (X, d, μ_{j+1}) with

$$\beta = \alpha = \frac{\mu(X)}{2N(r)K} \leq \frac{\mu_{j+1}(X)}{2N(r)}.$$

Thus, there exist two open sets A and D satisfying $A \subset D$,

$$\mu_{j+1}(A) \geq \alpha, \qquad \mu_{j+1}(D) \leq 2N(r)\alpha = \frac{\mu(X)}{K}, \qquad d(A, D^c) \geq 3r.$$

We define the couple (A_{j+1}, D_{j+1}) by

$$A_{j+1} = A \cap \left(\bigcup_{i=1}^{j} D_i\right)^c, \qquad D_{j+1} = D \cap \left(\bigcup_{i=1}^{j} D_i\right)^c.$$

It remains to check that the family $\{(A_1, D_1), \dots, (A_{j+1}, D_{j+1})\}$ satisfies the induction hypothesis. Indeed, the three first properties of this hypothesis are immediate consequences of the construction. To see that $d\big(A_{j+1}, \big(\bigcup_{i=1}^{j+1} D_i\big)^c\big) \geq 3r$ we only need to observe that

$$D = D_{j+1} \cup \left(D \cap \bigcup_{i=1}^{j} D_i\right) \subset \bigcup_{i=1}^{j+1} D_i,$$

which implies $d\big(A_{j+1}, \big(\bigcup_{i=1}^{j+1} D_i\big)^c\big) \geq d(A, D^c) \geq 3r$. □

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